Central Charges in Adjoint SQCD

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In order to understand the behavior of physical systems, it is often necessary to probe these systems at different length scales. When one views a system at progressively longer length scales, the number of massless degrees of freedom intuitively should decrease. The so-called c-theorem makes this statement more precise in two dimensions by stating that for a given system, there exists a function of coupling constants and length scale that monotonically decreases as the length scale increases. Although Zamolodchikov proved a c-theorem in two dimensions, finding an analogous theorem in four dimensions has proved to be problematic. A conjectured extension of the c-theorem to four dimensions has passed a variety of nontrivial tests, including one by Kutasov et al. for supersymmetric $SU(N_c)$ QCD with N_f flavors and an additional adjoint matter field, with $N_c, N_f \gg 1$. In this paper, we will demonstrate that Kutasov et al.'s result holds for the case where N_c and N_f are finite.

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1. Introduction

The renormalization group (RG) is a useful tool for describing physical systems at different energy scales. RG shows how the couplings of a theory flow from one scale to another. There is a general intuition that as you view a system at progressively greater length scales, the number of massless degrees of freedom decreases. For example, for a theory with a single free massive scalar, the effective number of massless of degrees of freedom goes from one at high energies to zero at low energies. RG always flows from high energy to low energy. An especially interesting feature of RG flows is that they can flow to fixed points. At fixed points, new symmetries appear, such as scale invariance and conformal invariance. As a result of this, a theory at one of these fixed points is said to be a conformal theory [1]. One has more calculational power when working with a theory that is conformal, because of the extended symmetry.

In two dimensions, the central charge c of a conformal field theory (CFT) is related to the number of degrees of freedom for the theory. For example, a free scalar has central charge $c = 1$. This quantity c decreases monotonically under RG flow; as the energy level decreases [1], the number of degrees of freedom decreases. Zamolodchikov's c-theorem proves that this statement always holds in two dimensions, but there is currently no analogous theorem in four dimensions. In four dimensions, there are two candidate central charges, a and c, for reasons we will explain below. In 1988, Cardy conjectured a four-dimensional analogue for the c-theorem, the "a-theorem" [2], which states that in the four-dimensional case, the central charge a should decrease monotonically under RG flow. Cardy made this conjecture based on a clever manipulation of the stress-energy tensor in four dimensions, which eliminated all candidate charges except for a. This is the reason Cardy's conjectured theorem focuses on a, rather than c [2]. There is no proof for the conjecture, so it is not a theorem, even though it is usually referred to as such in the literature. There have been many nontrivial checks for which the conjecture is true [4,5]. It has recently been shown that at least one counterexample to Cardy's conjecture exists [6], but in an overwhelming number of examples, the "a-theorem" works.

[1] In the units that we use, $\hbar = c = 1$, and energy is the inverse of length. As a result of this, RG can be viewed as flowing from high energy to low energy or as flowing to progressively greater distance scales

[2] It has since been proven that the central charge c can increase under RG flow in four dimensions [3].

In quantum field theories, coupling constants receive quantum corrections. Depending on the theory, these corrections can either make the theory more strongly or weakly coupled at low energies. Quantum chromodynamics (QCD), the type of theory inspected in this paper, happens to become strongly coupled at these low energies. The primary tool for doing calculations in a quantum field theory is perturbation theory, which expands in the aforementioned coupling constants. Perturbation theory is only possible if the coupling constants are small. These perturbations are achieved by introducing a small interaction to the theory, so that this interaction perturbs the theory away from the free theory [7]. This technique is not applicable to QCD since it is strongly coupled at low energy. Understanding low-energy QCD is still an open problem in physics. Although perturbation theory is not applicable to QCD, there are sometimes ways that calculations can be done in a theory that flows to strong coupling at low energy. It helps immensely if a theory is supersymmetric (SUSY); in other words, if every particle of integer spin has a related particle of half-integer spin, called a "superpartner." Thus, for a supersymmetric theory, for every fermion there exists a boson, and vice-versa. SUSY makes it easier to do calculations in a theory, since the more symmetries a theory has, the easier it is to make calculations in the theory. If a theory such as QCD can be made supersymmetric and is also conformal (superconformal), we can do several nontrivial calculations in it.

One relevant quantity we can try to calculate is the exact scaling dimension of an operator, $\mathcal{O}$. The scaling dimension of an operator $\mathcal{O}$ describes how $\mathcal{O}$ scales as you change the energy scale of the problem. Quantum effects can change the classical scaling dimension, so these dimensions are typically very difficult to calculate. For superconformal theories, these scaling dimensions are related to a $U(1)$ charge called the R-symmetry, which exists in any superconformal theory. The nature of this symmetry is discussed later in the paper. Intriligator and Wecht developed a technique called a-maximization [8], which is utilized later in this paper. a-maximization is a technique for finding the R-symmetry of a theory, utilizing the fact that the correct R-charges will maximize a cubic expression for a. Once these charges are calculated, it is then possible to calculate a, the four-dimensional central charge [5]. As a result of this, for a superconformal theory, it is actually possible to check the conjectured a-theorem for a nontrivial case.

One useful SUSY theory is the SUSY analogue of QCD, called supersymmetric quantum chromodynamics (SQCD). We will take the gauge group in this theory to be $SU(N_c)$, and add N_f flavors of quarks. This generalizes actual QCD, where there are three colors, red, green, and blue, thus $N_c = 3$. In actuality, these colors simply signify three different types of charge. In actual QCD, there are also six flavors, up, down, charm, strange, top, and bottom, and thus $N_f = 6$. So the gauge group $SU(N_c)$ with N_f flavors is a generalization of the gauge group $SU(3)$ with 6 flavors of quarks. However, since SQCD is especially simple, the R-charges are uniquely determined by the symmetry, and a-maximization is not necessary. The case where we add an adjoint matter field to SQCD is more interesting, and was studied by Kutasov et al. in 2003 [4]. Through a-maximization, they were able to calculate the R-charges for this theory, calculate the central charge, a, and verify that for SQCD with an adjoint field, the conjectured a-theorem holds. This was done by giving masses to flavors in the calculated expression for a, reducing the massless degrees of freedom. Since a with respect to N_f proved to be a monotonically increasing function, the "a-theorem" held. However, Kutasov et al. worked in the case where N_c, $N_f \gg 1$. It is not absolutely clear if the "a-theorem" will hold hold for the case where N_c, N_f are finite. In this paper, we check this case.

The outline of this paper is as follows. In Section 2, we will briefly discuss the renormalization group and review its fundamental concepts. These include Kadanoff's block spin derivation and the β-function. Section 3 contains a discussion of Zamolodchikov's c-theorem in two dimensions, and briefly discusses the reasons that it is not possible to translate Zamolodchikov's theorem directly to the four-dimensional case. Section 4 contains a brief overview of Cardy's conjectured four-dimensional a-theorem. Section 5 introduces Intriligator and Wecht's a-maximization technique for calculating the central charge a of a superconformal theory. Using this technique, we will replicate Kutasov et al.'s result for large N_f, N_c in SQCD with an adjoint matter field, also known as Adjoint SQCD. In Section 6, we will repeat this procedure for the finite N_f, N_c case and calculate a. Once this quantity is calculated, we can check that a decreases monotonically under RG flow. In Section 7, we conclude.

2. Renormalization Group Flow And Degrees of Freedom

In order to investigate the manner in which quantities change along RG flow, it is first necessary to have an intuitive understanding of RG flow itself. RG flow is an essential tool that is needed in order to discuss the c-theorem and its four-dimensional analogue, the "a-theorem." One of the most accessible models of RG flow was developed by Kadanoff in 1966 [9]. The paper examines the Ising model of spins on a d-dimensional hypercubic lattice with lattice spacing a. Let S_i represent some spin at site i in the hypercubic lattice. The Hamiltonian $\mathcal{H}$ of the system is then

$$\beta\mathcal{H} = -\beta J \sum_{\langle ij \rangle = 1} S_i S_j - \beta H \sum_i S_i. \tag{2.1}$$

In the above, S_i and S_j are arbitrary spins of value $+1$ or -1. J and H quantify contributions to the energy of the system. J corresponds to the interaction of neighboring spins through a ferromagnetic ($J > 0$) or antiferromagnetic ($J < 0$) coupling. H corresponds to an external magnetic field, and the term proportional to H in (2.1) corresponds to the interactions of the spins with this magnetic field. The quantity $\beta = \frac{1}{k_B T}$, where k_B is Boltzmann's constant and T is the temperature. If we define $K \equiv \beta J$ and $h \equiv \beta H$, then we can say that the Hamiltonian of the system is expressed in terms of the two state variables K and h. Now, through a course-graining procedure, we can replace all spins within a block of side la by a single spin called a block spin, where $l > 1$ is used to rescale the system. This block actually contains l^d spins, and its spin is the average of the spins inside. The procedure is effectively a rescaling of the system. This coarse-graining procedure is known as the block spin transformation [10].

We take the original coupling constants K and h to correspond to the value $l = 1$. Under the assumption that the Hamiltonian of the block spin is of the same form as that of the original Hamiltonian, the Hamiltonian when the system is viewed at $l > 1$ will very strongly resemble (2.1), but there will be fewer degrees of freedom, i.e. spins. K and h change under the block spin transformation and become functions of l. This creates a new Hamiltonian with modified K and h. In other words, there are new couplings at new scales. For example, the coupling h is replaced by a new expression, h_l. h_l is dependent on l. The expression for h_l is

$$h_l = h\bar{m}_l l^d. \tag{2.2}$$

In the above equation, $\bar{m}_l$ represents the average magnetization of the block in the block spin transformation, which also has a dependence on the value l. This average magnetization comes from a normalization of the spin S_I of a block, expressed as

$$\begin{aligned} S_I &\equiv \frac{1}{|\bar{m}_l|} \frac{1}{l^d} \sum_{i \epsilon I} S_i \\ \bar{m}_l &\equiv \frac{1}{l^d} \sum_{i \epsilon I} \langle S_i \rangle. \end{aligned} \tag{2.3}$$

Using this quantity when rescaling the magnetic field h, we find that

$$h\sum_i S_i = h\bar{m}_l l^d \sum_I S_I \equiv h_l \sum_I S_I. \tag{2.4}$$

Utilizing (2.3) and (2.4), we arrive at (2.2). The coupling constant K, which represents the interaction of the neighboring spins, also develops a dependence on l. The new Hamiltonian is written as

$$-\beta\mathcal{H}_l = K_l \sum_{\langle IJ\rangle} S_I S_J + h_l \sum_{I=1} S_I. \tag{2.5}$$

In the above equation, I and J represent sites of the new block spins. This block spin transformation is one example of a renormalization group transformation. This type of transformation can be generalized to other types of systems. A renormalization group transformation represents a change of the coupling constants due to a change of scale, as well as a rescaling of the system's degrees of freedom. The renormalization group for the cases that we will examine flows from the ultraviolet to the infrared. The ultraviolet (UV) is defined as the regime in which the theory is at high energy, while the infrared (IR) is the regime in which the theory is at low energy, in analogy with electromagnetic radiation.

In order to take something like the block spin transformation, which is a discrete rescaling of the system, and express it as a continuous rescaling of parameters, it is necessary to introduce the β-function. The β-function represents the flow of couplings at different energy scales. The β of the β-function is unrelated to the β used in (2.1). Generally, the β-function can be expressed as

$$\beta(g) = \mu\frac{\partial g}{\partial \mu}. \tag{2.6}$$

In the above, g is any one coupling parameter of the system, and μ represents the energy scale of the system [11]. The utility of RG extends beyond Ising models and block spin transformations; RG flow and β-functions can also be applied to quantum field theories, such as SQCD. In the block spin transformation, the β-function is used to represent the flow of the coupling constants: the ferromagnetic interaction and magnetic interaction. In other words, it measures how their strength varies under RG flow. The β-function can also represent the flow of couplings for quantum field theories in the same way. For example, consider $SU(N_c)$ SQCD with N_f flavors. The β-function is

$$16\pi^2\beta(g) = -b_0 g^3 + b_1 g^5 + \mathcal{O}(g^7) + ... \tag{2.7}$$

In (2.7), g is the gauge coupling and $\mathcal{O}$ denotes a term of the order g^7. In the above β-function, each term of higher order represents another "loop" of quantum correction. Each of these loops corresponds to an order in perturbation theory. The zero-loop case is simply the classical regime. Each loop after this represents a better approximation of the system's behavior, taking into account quantum mechanical effects. For the two loop beta function at small coupling, the g^3 and g^5 terms dominate the β-function, while higher loops of order g^7 or greater offer negligible contributions to the β-function. b_0 represents the one-loop component of the β-function, and more exactly,

$$b_0 = 3N_c - N_f, \tag{2.8}$$

while b_1 represents the two-loop component of the β-function. b_1 is also in terms of N_c and N_f, and is approximately proportional to N_c^2. This two loop β-function is a result from quantum field theory [11]. We will be working in the regime where b_0 is positive. In other words, $N_f < 3N_c$. b_1 will also always be positive in this regime.

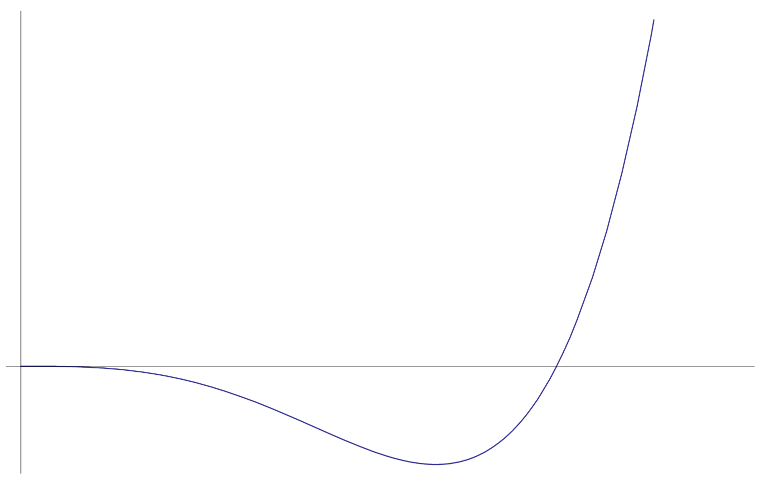

Figure 1: *A two loop β-function as a function of g.*

For small g, the β-function is dominated by the g^3 term, which is negative. For larger g, it is dominated by the the g^5 term, which is positive. This can be seen in the above graph. The renormalization group flow drives the β-function towards a fixed point, where the β-function is equal to zero. After infinite iterations of the renormalization group flow, it will inevitably flow towards such fixed points, although these fixed points might be trivial, i.e. free [10]. In the cases relevant to this investigation, the theory will flow to interacting fixed points in the IR. It is at these fixed points that we will use a-maximization to calculate the central charge a of a system, and subsequently check that the central charge decreases monotonically under RG flow, as Zamolodchikov's c-theorem predicts in two dimensions. But first, it is necessary to briefly discuss the c-theorem.

3. The c-theorem

In a two-dimensional CFT, the central charge c is a quantity that appears in numerous computations. For example, it is a coeffecient to the expected value of the two-point function of the stress-energy tensor T_{zz} [2]. The stress energy tensor is a symmetric (2,0) tensor that contains information about energy-like aspects of a system, such as energy density, pressure, and stress. The trace of the stress-energy tensor is $\theta \equiv T^{\mu}_{\mu} = T_z{}^z + T_{\bar{z}}{}^{\bar{z}}$, where z is the complex coordinate $z = x + iy$ and $\bar{z}$ is its complex conjugate, $\bar{z} = x - iy$. The exact relationship between the expected value of the two-point function of T_{zz} and c is

$$\langle T_{zz}(z) T_{zz}(0) \rangle = \frac{c}{2} \frac{1}{z^4} + \dots . \tag{3.1}$$

The central charge also appears in other computations. For example, c appears in the coeffecient of the conformal anomaly in the trace θ of the stress-energy tensor T_{zz}, which appears when a theory is placed on a curved background. This expression is

$$\langle \theta \rangle = -\frac{cR}{12}. \tag{3.2}$$

In the above, the normalized scalar curvature of the background the theory is placed on is represented by R [2].

The central charge is related to the number of degrees of freedom in a theory – the larger the central charge, the more degrees of freedom a theory has. In the case of a free field, the central charge of a scalar is $c = 1$, while the central charge of a spin $\frac{1}{2}$ fermion is $c = \frac{1}{2}$ [1]. In general, we will treat the central charge c as the number of degrees of freedom of a system, even when the system is interacting and the number of degrees of freedom can be difficult to determine. Since the number of massless degrees of freedom decreases along RG flow, the central charge should also decrease.

The c-theorem, derived by Zamolodchikov in 1986 [12], makes more precise the relationship that exists between the RG flow and the central charge c in two dimensions. An intuitive and enlightening summary of Zamolodchikov's c-theorem is given by Polchinski[3] [13]. The aim of Zamolodchikov's theorem is to define a function C that is equal to the central charge c at fixed points in the ultraviolet and infrared, and which monotonically decreases in between these two points. In order to do this, it is necessary to define three somewhat arbitrary functions:

$$F(r^2) = z^4 \langle T_{zz}(z,\bar{z}) T_{zz}(0,0) \rangle \tag{3.3}$$

$$G(r^2) = 4z^3 \bar{z} \langle T_{zz}(z,\bar{z}) T_{z\bar{z}}(0,0) \rangle \tag{3.4}$$

$$H(r^2) = 16 z^2 \bar{z}^2 \langle T_{z\bar{z}}(z,\bar{z}) T_{z\bar{z}}(0,0) \rangle . \tag{3.5}$$

In the above equations, T_{zz} is the ordinary stress-energy tensor, and its subscripts indicate two coordinates in two-dimensional space. Rather than express them in the normal two-dimensional plane with coordinates (x, y) and $(x, -y)$, it is convenient to express them in complex coordinates $(z = x + iy, \bar{z} = x - iy)$ so that $z\bar{z} = r^2$. The dependence of these functions solely on $z\bar{z} = r^2$ makes them rotationally invariant. By conservation, $\frac{\partial T_{z\bar{z}}}{\partial z} + \frac{\partial T_{zz}}{\partial \bar{z}} = 0$, so

$$\begin{aligned} &4\frac{\partial F(r^2)}{\partial r^2} + \frac{\partial G(r^2)}{\partial r^2} - 3G(r^2) = 0 \\ &4\frac{\partial G(r^2)}{\partial r^2} - 4G(r^2) + \frac{\partial H(r^2)}{\partial r^2} - 2H(r^2) = 0. \end{aligned} \tag{3.6}$$

From the functions $F(r^2)$, $G(r^2)$, and $H(r^2)$, it is possible to carefully construct a function, the Zamoldchikov c-function, which is

$$C = 2F - G - \frac{3}{8}H. \tag{3.7}$$

It might seem as if this linear combination of F, G, and H is arbitrary, but keep in mind that it is carefully constructed in such a way that the C-function is equal to the central charge c at RG fixed points. This is because at an RG fixed point, a theory is conformal. When a theory is conformal, the trace of the stress-energy tensor is equal to zero. More explicitly,

$$\theta = T_z{}^z + T_{\bar{z}}{}^{\bar{z}} = 2T_{z\bar{z}} = 0. \tag{3.8}$$

[3] Our discussion here uses this approach.

Given this relationship, the C-function simply simplifies to $C = 2F(r^2)$. Keeping in mind the expected value of the two-point function defined in (3.1), the expression simplifies as

$$C = 2F = 2z^4(\frac{c}{2}\frac{1}{z^4}) = c. \tag{3.9}$$

Using the relationships defined in (3.6), we also find that when differentiating this defined C-function with respect to r^2, it is possible to rewrite the result solely in terms of H,

$$\frac{\partial C(r^2)}{\partial r^2} = -\frac{3}{4}H(r^2). \tag{3.10}$$

The function H only depends on r^2, as the values of the stress-energy tensor are solely dependent on r^2, and the leading coeffecient to H is $16z^2\bar{z}^2 = 16r^4$. From this, we see that H is always positive, and thus, the function C decreases monotonically with longer and longer distance scales. Thus, by Zamolodchikov's c-theorem, it is possible to define a function C which can be used to calculate the central charge c at fixed points, and that monotonically decreases in between these RG fixed points.

While Zamolodchikov's theorem holds in two dimensions, the corresponding conjecture in four dimensions, the "a-theorem," has not been proven. In four dimensions, there are more potential objects that might describe the degrees of freedom of a system. When placing a theory on a curved background, as we did in (3.2), there are more terms containing coeffecients that are similar to c in the case where $d \neq 2$. As a result of this, more candidate central charges appear. In addition to c, there exists another term, a. Due to there being more than one central charge in the four-dimensional case, it is unclear which quantity, if any, would obey Zamolodchikov's c-theorem. Because of this, it is difficult to be sure which quantities to use in order to create a clear four-dimensional analogue to the c-theorem.

4. The a-theorem

In 1988, Cardy claimed that a was the four-dimensional analogue of the central charge c in two dimensions [2]. The quantity c is related to the expected value $\langle\theta\rangle$ of the trace of the stress-energy tensor. Cardy was able to find an expression for c by integrating $\langle\theta\rangle$ with constants over the two-dimensional sphere S^2 [2]. The exact relation is

$$c = -\frac{3}{\pi}\int_{S^2}\langle\theta\rangle\sqrt{g}\,d^2x. \tag{4.1}$$

Having defined this new method of calculating the central charge c, Cardy conjectured that by integrating in four dimensions over the sphere S^4, it might be possible to prove a four-dimensional analogue to the c-theorem. On both S^2 and S^4, the radius of the sphere is analogous to the scale of the system, so that the greater the radius, the greater the length scale of the system. In four dimensions, there is more than one candidate central charge. When integrating over S^4, all terms vanish except for the one proportional to the candidate central charge a. Through this manipulation, Cardy was able to propose a conjectured four-dimensional c-theorem, called the "a-theorem." The a-theorem conjectures that this quantity a decreases monotonically under RG flow, just as c does in two dimensions.

Four-dimensional quantum field theory is more complicated than two-dimensional quantum field theory, so this "theorem" has been difficult to prove. Calculations are made simpler when a theory has SUSY. As its name implies, a SUSY theory has more symmetries than a normal theory. These added symmetries make calculation easier. A 1998 paper by Anselmi et al. [14] found that in SUSY theories, the value of central charge candidates, such as a, is equal to the value of trace anomaly coefficients at IR fixed points, as we discuss. Using this relation, it is possible to find an expression for the central charge a in terms of the R-charges. Superconformal theories have a $U(1)$ symmetry called the R-symmetry. The R-symmetry assigns charges to every field in the theory. For example, a field ϕ with R-charge α transforms as $\phi \to e^{i\alpha}\phi$ under the R-symmetry [13]. Since the R-charge is a quantity associated with supersymmetric theories, this method is not valid for non-supersymmetric theories. Anselmi et al.'s paper claims that the central charge a of a supersymmetric theory can be calculated with the relation

$$a = \frac{3}{32}(3\mathrm{Tr}R^3 - \mathrm{Tr}R) \equiv \frac{3}{32}\hat{a}. \tag{4.2}$$

Written more explicitly, this relationship can also be expressed as

$$a = \frac{3}{32}(2\dim G + \sum_i (\dim R_i(1-r_i)(1-3(1-r_i)^2)). \tag{4.3}$$

These two expressions for the central charge a are equivalent. In the above, G represents the gauge group, and $\dim G$ represents the dimension of the gauge group. The term written as $2\dim G$ represents contributions to a from gauginos in the theory. Gauginos are the superpartners to the gauge bosons that are required by supersymmetry [13]. The summation over i represents contributions from different fields R_i in the theory, summed over the number of flavors in that field. The value r_i indicates the R-charge of any given field. The reason that this equation contains terms proportional to $(r_i - 1)$ and not r_i is because these terms are the R-charge of the fermionic component of the field R_i, rather than the bosonic component [14] [4].

Following this paper, Intriligator and Wecht [8] developed a technique called "a-maximization." The a-maximization technique uses the relationship for the central charge a derived by Anselmi et al. and states that the correct R-charge R of a theory uniquely determines the central charge a. By maximizing this expression with respect to R, it is possible to find the value of the central charge a.

5. Reproducing the findings of Kutasov et al.

A 2003 paper by Kutasov et al. [4] utilizes a-maximization in order to check that for quarks and gluons with an adjoint matter field, the central charge a decreases under RG flow. The theory under consideration has gauge group $SU(N_c)$ with N_f flavors. There is also an adjoint field present. The theory is supersymmetric. SQCD is more user-friendly than QCD because of its supersymmetry; with more symmetries, it is possible to make calculations in the theory that would not be possible in normal QCD. SQCD's R-charges are determined by SQCD's symmetries alone, by virtue of the fact that it is such a simple theory. Because of this, we instead consider SQCD with an adjoint matter field. A table of relevant quantities in adjoint SQCD can be found below.

Field	Representation	Dimension	Number of Flavors	$T(r_i)$
Q	N_c	N_c	N_f	1
$\tilde{Q}$	$\bar{N}_c$	N_c	N_f	1
X	Adjoint	${N_c}^2 - 1$	1	$2N_c$

(5.1)

$T(r_i)$ is the quadratic Casimir corresponding to a field's representation, a relevant QFT quantity [4]. The values of $T(r_i)$ and dimension for any given field can be found in e.g Peskin and Schroeder [11], while the field, representation, and number of flavors are part of the definition of a given field. The first step in Kutasov et al.'s calculation is to use the anomaly-free equation of an R-symmetry, which states that

$$T(G) - \sum_i T(r_i)(r_i - 1) = 0. \tag{5.2}$$

[4] This is because the R-symmetry does not commute with supersymmetry.

(5.2) is proportional to the β-function for the theory [15]. At a non-trivial fixed point, this quantity must equal zero, as the definition of an RG fixed point indicates. $T(G)$ is the value of $T(r_i)$ when the r_i is adjoint, and can also be found in a reference table. The quantity $T(G)$ depends only on the gauge group, not the representation of the matter in the theory. The summation in the expression is a sum over all matter in the theory. For the example in question, we are dealing with three different types of R – that of quarks and their antiparticles, and that of the adjoint field X. We will represent these charges by $R(Q)$, $R(\tilde{Q})$ and $R(X)$. $R(Q) = R(\tilde{Q})$, by charge conjugation symmetry. Using (5.2), it is possible to rewrite $R(X)$ in terms of $R(Q)$. Taking into account contributions from the Q, $\tilde{Q}$, and X fields, we find that the anomaly free equation for the theory is

$$2N_c + 2N_f(R(Q) - 1) + 2N_c(R(X) - 1) = 0. \tag{5.3}$$

Using this equation we can rewrite $R(X)$ in terms of $R(Q)$, N_c, and N_f. If like Kutasov et al. we rename $R(Q)$ and $R(\tilde{Q})$ y for convenience, then we arrive at

$$R(X) = \frac{N_f - N_f R(Q)}{N_c} = \frac{N_f}{N_c}(1 - R(Q)) \equiv \frac{N_f}{N_c}(1 - y). \tag{5.4}$$

Having calculated the relationship between $R(Q)$ and $R(X)$, the next step is to find an expression for $\hat{a}$ entirely in terms of $R(Q)$. Once this is done, we can use a-maximization to find the value of $R(Q)$ that maximizes a. Using the equivalence between (4.2) and (4.3), it is possible to find the contributions to the central charge a from $R(Q)$, $R(\tilde{Q})$, and $R(X)$. The contribution from $R(X)$ is

$$\begin{gathered}({N_c}^2 - 1)\left[(R(X) - 1)^3 - (R(X) - 1)\right] = \\ ({N_c}^2 - 1)\left[\left(\frac{N_f}{N_c}(1 - y) - 1\right)^3 - \left(\frac{N_f}{N_c}(1 - y) - 1\right)\right].\end{gathered} \tag{5.5}$$

The contribution from $R(Q)$ or $R(\tilde{Q})$ is

$$N_f N_c \left[(R(Q) - 1)^3 - (R(Q) - 1)\right] = N_f N_c \left[(y - 1)^3 - (y - 1)\right]. \tag{5.6}$$

Kutasov et al. do the calculation in the regime where $N_c \gg 1$ and $N_f \gg 1$. Due to this, the $({N_c}^2 - 1)$ term in (5.5) becomes simply ${N_c}^2$. Making this approximation and putting the calculated contributions from $R(X)$, $R(Q)$, and $R(\tilde{Q})$ back into expression (4.2)for a, we are able to replicate Kutasov et al.'s result, which states that

$$\hat{a} = 6N_fN_c(y-1)^3 - 2N_fN_c(y-1) + 3{N_c}^2\left(\frac{N_f}{N_c}(1-y)-1\right)^3 - {N_c}^2\left(\frac{N_f}{N_c}(1-y)-1\right) + 2{N_c}^2. \tag{5.7}$$

The term $2{N_c}^2$ comes from gauginos. They must appear in the adjoint representation of the gauge group. Quantitatively, they are represented in (4.3) by the quantity $2\dim G$.

While this expression for central charge a is correct, a-maximization requires that we find the central charge solely in terms of N_c and N_f. Notice that this representation of a is with respect to N_c, N_f, and y. This can be remedied by maximizing the equation with respect to y, and thus finding an expression for y in terms of N_c, N_f. Differentiating a with respect to y, we find that

$$\frac{\partial \hat{a}}{\partial y} = 18N_fN_c(y-1)^2 - 2N_fN_c - 9N_fN_c\left(\frac{N_f}{N_c}(1-y)-1\right)^2 + N_cN_f. \tag{5.8}$$

Setting $\frac{\partial \hat{a}}{\partial y}$ equal to zero can we find the value of $y = R(Q)$ that maximizes a. This is the correct value for $R(Q)$. While our initial result was different from Kutasov et al.'s result, it can be manipulated in such a way as to show that it is in fact the same. The value of $R(Q)$ that maximizes $\hat{a}$ is

$$R(Q) = y = \frac{3 + \frac{N_c}{N_f}\left(-3 - 6\frac{N_c}{N_f} + \sqrt{20\frac{{N_c}^2}{{N_f}^2} - 1}\right)}{3 - 6\frac{{N_c}^2}{{N_f}^2}}. \tag{5.9}$$

By substituting the calculated value of $R(Q)$ into (5.4), the R-charge $R(X)$ that is contributed by the adjoint matter field follows from simple algebra. Placing these values of $R(Q)$ and $R(X)$ back into (5.7), we an expression for a purely in terms of N_c and N_f. Our expression for $\hat{a}(N_f, N_c)$ is

$$\hat{a}(N_f, N_c) = \frac{50N_c^2\left(40{N_c}^2 + {N_f}^2\left(7 + 9\sqrt{20\frac{{N_c}^2}{{N_f}^2} - 1}\right)\right)}{9\left(10{N_c}^2\left(9 + \sqrt{20\frac{{N_c}^2}{{N_f}^2} - 1}\right) + 9 + 13{N_f}^2\sqrt{20\frac{{N_c}^2}{{N_f}^2} - 1}\right)}, \tag{5.10}$$

while Kutasov et al.'s expression for $\hat{a}(N_f,\, N_c)$ is

$$\hat{a}(N_f, N_c) = \frac{2{N_c}^2}{9(1 - 2\frac{{N_c}^2}{{N_f}^2})^2}\left(18 - \left(20\frac{{N_c}^2}{{N_f}^2} - 1\right)^{\frac{3}{2}} - 90\frac{{N_c}^2}{{N_f}^2}\right). \tag{5.11}$$

While the result derived in this paper may at first seem to be different from that found by Kutasov et al., it is not. Evaluation of $\hat{a}$ for various values of N_c, N_f shows that the two expressions for $\hat{a}$ are in fact the same. The graph of $\hat{a}$ as a function of N_f for the value $N_c = 50$ can be found below. It is not expected that the theory is conformal all the way down to $N_f = 0$.

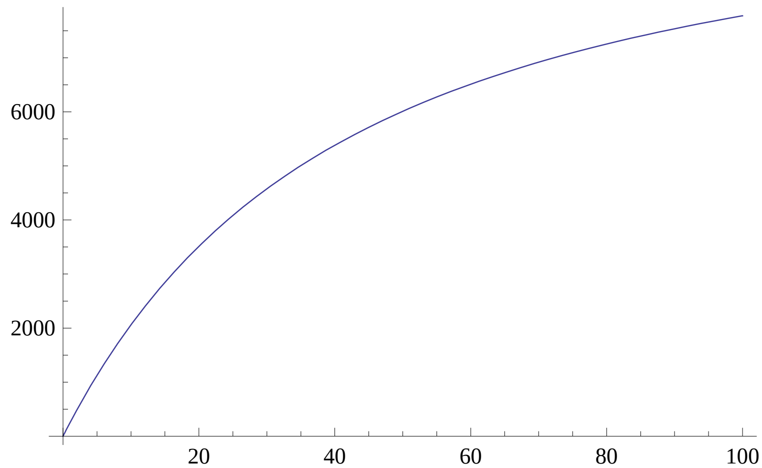

Figure 2: *$\hat{a}$ as a function of N_f for Adjoint SQCD in the regime where N_c, $N_f \gg 1$. $N_c = 50$*

5.1. Calculating $\frac{\partial \hat{a}}{\partial N_f}$

In order to check that SQCD with an adjoint matter field obeys Cardy's conjectured a-theorem, it is necessary to check that a as a function of N_f is monotonically increasing. Physically, the flow is to decreasing N_f, but computationally our approach is equivalent. As we increase N_f, we can imagine that we are adding massless quarks to the theory. Alternatively, we can imagine that as we decrease N_f we are giving masses to quarks in the theory, and then flowing below that mass scale. According to the conjectured a-theorem in four dimensions, this increase in massless degrees of freedom should correspond with an increase in the value of central charge a. Thus, according to Cardy's conjectured a-theorem, $\frac{\partial \hat{a}}{\partial N_f}$ should always be positive.

In order to do check that a increases monotonically with respect to N_f for SQCD with an adjoint matter field, let's look at the expression for a that was derived previously in (5.11). Differentiating with respect to N_f, we find that

$$\frac{\partial \hat{a}}{\partial N_f} = \frac{2{N_c}^2\left(180\frac{{N_c}^2}{{N_f}^3} - 60\frac{{N_c}^2}{{N_f}^3}\sqrt{20\frac{{N_c}^2}{{N_f}^2}-1}\right)}{9\left(1-2\frac{{N_c}^2}{{N_f}^2}\right)^2} - \frac{16{N_c}^4\left(18+\left(20\frac{{N_c}^2}{{N_f}^2}-1\right)^{\frac{3}{2}} - 90\frac{{N_c}^2}{{N_f}^2}\right)}{9\left(1-2\frac{{N_c}^2}{{N_f}^2}\right)^3{N_f}^3}. \tag{5.12}$$

While it is not immediately apparent that $\frac{\partial \hat{a}}{\partial N_f}$ is > 0 for all values of N_f, inspection of the graph of $\frac{\partial \hat{a}}{\partial N_f}$ shows that the value of the derivative is never less than zero. Accordingly, it seems that the central charge a increases monotonically with respect to N_f in the large N_c, N_f limit, as Kutasov et al. calculated. Thus, in the N_c, $N_f \gg 1$ regime, SQCD with an adjoint matter field obeys Cardy's conjectured a-theorem.

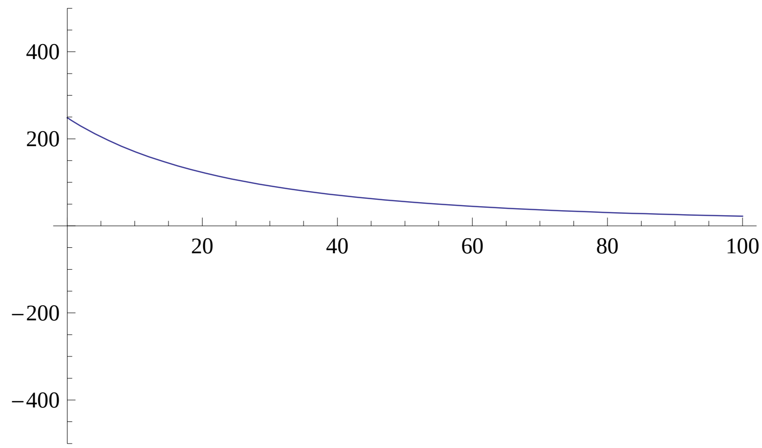

Figure 3: *$\frac{d\hat{a}}{dN_f}$ as a function of N_f for Adjoint SQCD in the regime where N_c, $N_f \gg 1$. $N_c = 50$*

6. Reproducing the findings of Kutasov et al. for finite N_c, N_f.

What the calculation by Kutasov et al. shows is that for SQCD with an adjoint matter field, Cardy's conjectured a-theorem holds. $a_{UV} > a_{IR}$, where the UV is the theory with N_f quarks and the IR is the theory with $N_f - 1$ quarks. However, Kutasov et al.'s calculation only shows this for N_c, $N_f \gg 1$. The case in which both of these values is finite has yet to be calculated. Through small adjustment of the above calculations, one can easily check as to whether this conjectured a-theorem holds for finite N_c, N_f.

Large portions of the work done by Kutasov et al. can simply be transplanted to the case where both N_c and N_f are finite. The anomaly free equation of an R-symmetry can still be used to determine $R(X)$ in terms of $R(Q)$. Also, all of the data contained in (5.1) is still applicable to the fields in the case of large N_c, N_f. However, there is one crucial difference in the calculation.

Recall that by (4.2), we can express the central charge a of SQCD with an adjoint matter field as

$$\hat{a} = 6N_f N_c(y-1)^3 - 2N_f N_c(y-1) + 3({N_c}^2 - 1)\left(\frac{N_f}{N_c}(1-y) - 1\right)^3 - ({N_c}^2 - 1)\left(\frac{N_f}{N_c}(1-y) - 1\right) + 2({N_c}^2 - 1), \tag{6.1}$$

taking into account contributions from quarks and their antiparticles, as well as contributions from the adjoint matter field. In this expression, as in that of Kutasov et al., $R(Q) = R(\tilde{Q})$ have been renamed y for convenience, and $R(X) = \frac{N_f}{N_c}(1-y)$. In the calculations by Kutasov et al., the term $({N_c}^2 - 1)$, which appeared as the dimension of the representation of the adjoint field, as well as the contribution to the charge a by the gauginos, was approximated to ${N_c}^2$ since $N_c \gg 1$. For our finite N_c, N_f calculation, this approximation cannot be made. Thus, the central charge a remains in the form (6.1)

Now that we have an expression for a in the regime where both N_c and N_f are finite, we can run the same tests on the expression that Kutasov et al. did in the large N_c, N_f case to test that a decreases monotonically under RG flow. Utilizing (6.1) as our expression for a, we can calculate $\frac{d\hat{a}(y)}{dy}$ for this new case. $\frac{d\hat{a}(y)}{dy}$ for the finite N_c, N_f case is

$$\frac{\partial \hat{a}}{\partial y} = -2N_cN_f + \frac{N_f({N_c}^2-1)}{N_c} - \frac{9N_f({N_c}^2-1)\left(\frac{N_f(1-y)}{N_c}-1\right)^2}{N_c} + 18N_cN_f(y-1)^2. \quad (6.2)$$

By maximizing this new expression for a by setting its derivative equal to zero, we find the value of y that maximizes a. Consequently, $R(Q)$ is

$$R(Q) = y = \frac{6{N_c}^4 - 3N_cN_f + 3{N_c}^3N_f + 3{N_f}^3 - 3{N_c}^2{N_f}^2 - \sqrt{20{N_c}^8 - 16{N_c}^6 - {N_c}^6{N_f}^2 + {N_c}^2{N_f}^2}}{3(2{N_c}^4 + {N_f}^2 - {N_c}^2{N_f}^2)}. \quad (6.3)$$

Again using relation (5.4), we find an expression for $R(X)$, also in terms of N_c and N_f, that follows from simple algebra. By substituting these values of $R(Q)$ and $R(X)$ back into (6.1), an expression for a is now derived entirely in terms of N_c and N_f for the case where N_c and N_f are both finite. Unfortunately, it is not as tidy as the expression for the large N_c, N_f approximation. The expression for $\hat{a}$ is

$$\hat{a}(N_c, N_f) = u + \frac{v}{w}, \quad (6.4)$$

where

$$u = 2N_f(-90{N_c}^9N_f - 9N_c{N_f}^3 + 36{N_c}^3{N_f}^3 + 18{N_c}^7N_f(9+{N_f}^2) - 9{N_c}^5N_f(8+5{N_f}^2)$$
$$v = (20{N_c}^6 + {N_f}^2 - {N_c}^4(16+{N_f}^2))\sqrt{{N_c}^2(20{N_c}^6 + {N_f}^2 - {N_c}^4(16+{N_f}^2))}$$
$$w = (9N_c(2{N_c}^4 + {N_f}^2 - {N_c}^2{N_f}^2)^2). \quad (6.5)$$

The graph of $\hat{a}$ as a function of N_f for the value $N_c = 5$ can be found below.

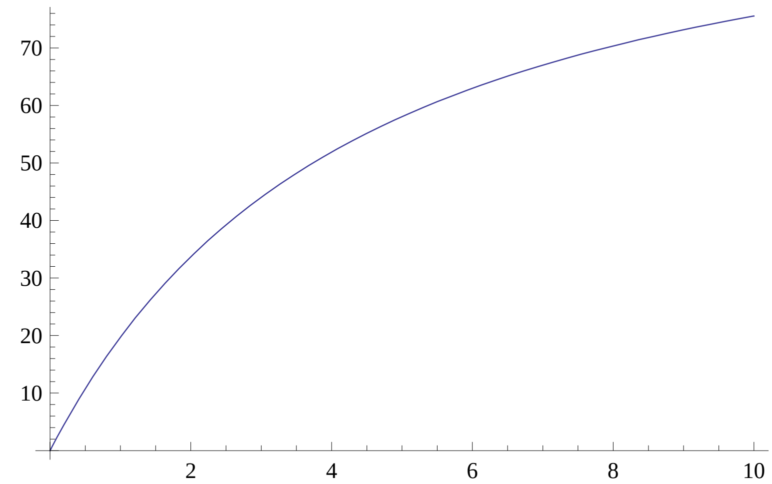

Figure 4: *$\hat{a}$ as a function of N_f for Adjoint SQCD in the finite N_c, N_f regime. $N_c = 5$*

To verify that this expression for $\hat{a}$ in the case where N_c and N_f is correct, it is helpful to check that it agrees with the expression for $\hat{a}$ that uses the N_c, $N_f \gg 1$ approximation at large N_c, N_f. The two functions can be seen graphed against each other below. The finite N_c, N_f case is the lower line, while the N_c, $N_f \gg 1$ case is the upper line. As N_f becomes increasingly large, the graphs become increasingly similar. Note that although the finite N_c, N_f function appears to blow up as it approaches $N_f = 70$, the theory is not expected to be conformal in this region. Consequently, these results remain valid.

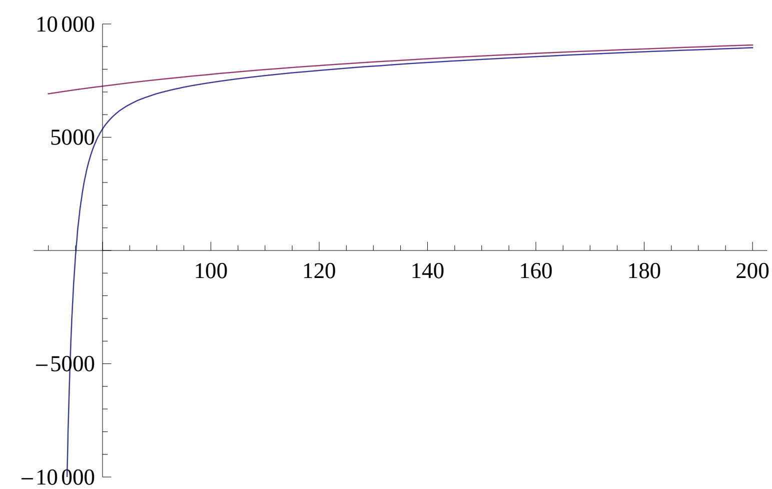

Figure 5: *$\hat{a}$ as a function of N_f for Adjoint SQCD for both large and finite N_c, N_f. $N_c = 50$*

6.1. Calculating $\frac{\partial \hat{a}}{\partial N_f}$

As in replicating the findings of Kutasov et al., in order to check that SQCD with an adjoint field obeys Cardy's conjectured a-theorem for finite N_c, N_f, it is necessary to check that $\hat{a}$ increases monotonically with respect to N_f. As in the N_c, $N_f \gg 1$ case, as one increases N_f, one can imagine that we are adding massless degrees of freedom to the theory. Thus, if the theory does in fact obey the conjectured a-theorem, this increase in degrees of freedom should bring about an increase in $\hat{a}$.

In analogy to the previous case, we check that the derived expression for $\hat{a}$ for SQCD with an adjoint matter field in the finite N_c, N_f limit increases monotonically with respect to N_f by taking the derivative of this expression for $\hat{a}$ and checking that it is always positive. Differentiating with respect to N_f, we find that

$$\frac{\partial \hat{a}(N_c, N_f)}{\partial N_f} = -8{N_c}^4 \left(\frac{m}{n} + p\right), \tag{6.6}$$

where

$$\begin{aligned} m &= -200{N_c}^{13} - 14N_c{N_f}^4 + 27{N_c}^3{N_f}^4 + 10{N_c}^{11}(32 - 25{N_f}^2) + {N_c}^5{N_f}^2(232 + {N_f}^2) \\ &\quad - {N_c}^7{N_f}^2(722 + 27{N_f}^2) + {N_c}^9(-128 + 740{N_f}^2 + 13{N_f}^4) \\ n &= \left[9(2{N_c}^4 + {N_f}^2 - {N_c}^2{N_f}^2)^3\sqrt{{N_c}^2\left(20{N_c}^6 + {N_f}^2 - {N_c}^4(16 + {N_f}^2)\right)}\right] \\ p &= \frac{90{N_c}^8N_f - 18{N_f}^3 + 45{N_c}^2{N_f}^3 + 9{N_c}^6N_f(-18 + {N_f}^2) - 36{N_c}^4N_f({N_f}^2 - 2)}{9(2{N_c}^4 + {N_f}^2 - {N_c}^2{N_f}^2)^3}. \end{aligned} \tag{6.7}$$

Inspection of the graph of this function with respect to N_f shows that $\frac{\partial \hat{a}(N_c, N_f)}{\partial N_f}$ is always greater than zero. In other words, the function $\hat{a}(N_f, N_c)$ is increasing monotonically. Thus, it seems that the result in the finite N_c, N_f limit is the same as that in the limit where N_c, $N_f \gg 1$. The central charge a monotonically increases with respect to N_f for SQCD with an adjoint matter field. Therefore, $a_{UV} > a_{IR}$ and the theory obeys Cardy's conjectured a-theorem for all N_c, N_f.

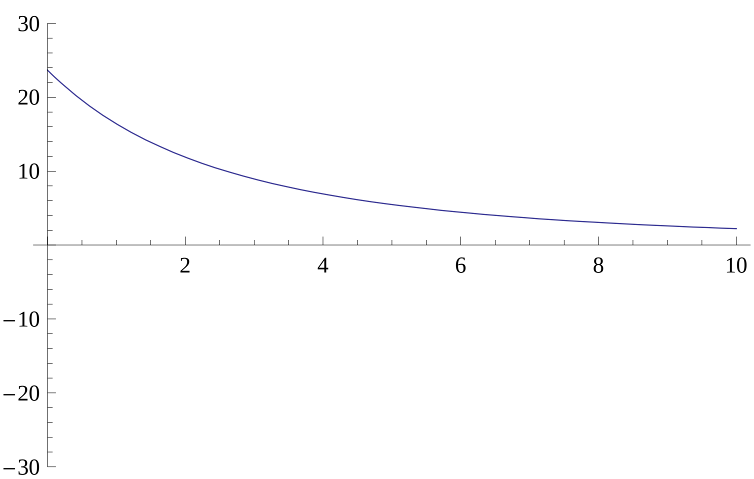

Figure 6: $\frac{\partial \hat{a}(N_c, N_f)}{\partial N_f}$ *as a function of* N_f *for Adjoint SQCD in the finite* N_c, N_f *regime.* $N_c = 5$

7. Conclusion

In this paper, it has been verified that Kutasov et al.'s calculations for SQCD with an adjoint matter field hold. The central charge a decreases monotonically under RG flow in the case where N_c, $N_f \gg 1$. This is checked by removing flavors from the theory by giving them a mass and flowing below the mass scale, thus removing massless degrees of freedom from the theory. We first introduced the basics of the renormalization group, and looked at how these principles related to quantum field theory. We then briefly discussed Zamolodchikov's c-theorem, and then discussed Cardy's conjectured a-theorem. We used these principles in order to replicate a nontrivial test of the conjectured a-theorem done by Kutasov et al.

Kutasov et al. tested that the conjectured a-theorem held for SQCD with an adjoint field in the limit where N_c, $N_f \gg 1$. This was done by removing flavors from the theory by giving them mass, and removing massless degrees of freedom from the theory. This result was reproduced by solving for a in terms of the R-charges of the theory using Intriligator and Wecht's a-maximization technique. By checking that the derivative of a with respect to N_f was always increasing, it was shown that $a_{UV} > a_{IR}$, as Cardy's conjectured a-theorem predicts.

This prediction was then extended. SQCD with an adjoint matter field was still the theory under investigation, but rather than considering the theory in the N_c, $N_f \gg 1$ limit, the same calculations were carried out in the regime where N_c and N_f were finite. This calculation was also done using a-maximization. As expected, this calculation verified that the central charge a of the theory also decreased monotonically under RG flow in the finite N_c, N_f limit.

There are slight problems with the a-maximization technique. These problems are discussed by Kutasov et al. in their paper [4]. The first of these problems is that there might be an accidental symmetry in a theory in the IR. Occasionally, these symmetries cause operators to violate a unitary bound. Kutasov et al. address these concerns in their paper, and find that the calculation for the N_c, $N_f \gg 1$ case still hold. There is no reason to believe that the same is not true for the finite N_c, N_f case. Further work on this calculation might indeed check these caveats to make this additional evidence for Cardy's a-theorem stronger.

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